Linear Algebra & Geometry

LECTURE 7

- Subspaces of a vector space
- Linear combinations
- Span

Definition.

Let V be a vector space over $\mathbb{K}$. A subset $W \subseteq V$ is called a *subspace* of V if W is a vector space over $\mathbb{K}$ under the same operations of vector addition and scalar multiplication, restricted to W.

Fact.

W is a subspace of V if and only if

1. $(\forall \alpha \in \mathbb{K})(\forall w \in W)\alpha \cdot w \in W$
2. $(\forall w_1, w_2 \in W)\ w_1 + w_2 \in W$
3. $W \neq \emptyset$ (or, equivalently $\Theta_V \in W$)

Proof.

The "only if" part: trivial. The only thing worth doing is $W \neq \emptyset$ is equivalent to $\Theta_V \in W$ (if 1 and 2 hold). Suppose $W \neq \emptyset$, *i.e.* there is a vector $w \in W$. Due to 1., $(-1)w \in W$ and, due to 2., $\Theta = w + (-1)w \in W$. The other implication is trivial, $\Theta_V \in W$ obviously means $W \neq \emptyset$.

The "if" part: 1.,2. and 3., together with last lecture theorem imply that $(W, +)$ is a subgroup of $(V, +)$ and that it is closed under scaling. The remaining axioms of a vector space follow from the simple fact that all vectors of W belong to V, which means they satisfy all required identities.

Examples. (subspaces)

Decide which of the following subsets are subspaces:

1. $\{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
2. $\{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
3. $\{(x, y) \in \mathbb{R}^2 : x = 5y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
4. $\{(x, y) \in \mathbb{R}^2 : x^2 = y\}$ in $\mathbb{R}^2$ over $\mathbb{R}$
5. $\{(x, y, z) \in \mathbb{R}^3 : x + y - 3z = 1\}$ in $\mathbb{R}^3$ over $\mathbb{R}$
6. $\left\{\{a, b\}, \{a\}, \varnothing\right\}$ in $2^{\{a,b,c\}}$ over $\mathbb{Z}_2$
7. The set of all finite sets from $2^{\mathbb{N}}$ over $\mathbb{Z}_2$

Comprehension self-test.

Find all subspaces of $\mathbb{R}^2$ over $\mathbb{R}$ (with usual operations).

Definition

Let V be a vector space over a field $\mathbb{K}$, let $a_1, \dots, a_n \in \mathbb{K}$ and $v_1, \dots, v_n \in V$. The vector $a_1 v_1 + \cdots + a_n v_n$ is called the *linear combination* of vectors $v_1, \dots, v_n$ with coefficients $a_1, \dots, a_n$.

A common problem in linear algebra is to decide whether a given vector is or is not a linear combination of other given vectors.

Example.

You fail Linear Algebra, and you decide to blow-up the MiNI Building. A recipe on the darknet says you mixing 30% of ingredient A, 50% of B and 20% of C will provide an explosive. A leading branch of toothpaste T consists of 10, 60 and 30 percent of those, a scouring powder S has 5, 80 and 15 and a washing machine powder P has 25, 50 and 25. Can you get your explosive mixing those in any proportion?

In the language of vectors we are asking if there exist coefficients t, s ,p such that

$$(30,50,20) = t(10,60,30) + s(5,80,15) + p(25,50,25)$$

i.e., is (30,50,20) is a linear combination of (10,60,30), (5,80,15) and (25,50,25).

In this example we must also require the coefficients to be ≥0.

Comparing the left-hand side to the right-hand side of

(30,50,20) = t(10,60,30) + s(5,80,15) + p(25,50,25)

component-by-component we clearly get a system of equations

$$\begin{cases} 30 = 10t + 5s + 25p \\ 50 = 60t + 80s + 50p \\ 20 = 30t + 15s + 25p \end{cases}$$

We can phrase our problem slightly differently: does our vector (30,50,20) belong to the set of all possible linear combinations of (10,60,30), (5,80,15) and (25,50,25).

No, we cannot blow-up Polytechnica. We cannot get 30% of A in our mixture because in each of the household chemicals the content of A is below 30%.

Definition

Let V be a vector space over a field $\mathbb{K}$ and let $S \in V$ be a set of vectors. The *span of the set S* we mean the set $span(S) \subseteq V$ defined by

$span(S) = \{a_1 v_1 + \ldots + a_n v_n | n \in \mathbb{N} \wedge (\forall i)(a_i \in \mathbb{K} \wedge v_i \in S)\}$ if $S \neq \emptyset$ and

$span(S) = \{\Theta\}$ if $S = \emptyset$.

In plain language, $span(S)$ is the set of all possible linear combinations of vectors from S.

Example.

Consider $\mathbb{R}^3$ and two vectors $(1, 2, 0), (0, 0, 3) \in \mathbb{R}^3$.

$span(\{(1, 2, 0), (0, 0, 3)\}) = \{a(1, 2, 0) + b(0, 0, 3) | a, b \in \mathbb{R}\} =$

$\{(a, 2a, 3b) | a, b \in \mathbb{R}\} = \{(x, 2x, y) | x, y \in \mathbb{R}\}$

Theorem.

Let V be a vector space over a field $\mathbb{K}$ and let S be a subset of V. Then W $= span(S)$ is a subspace of V.

Proof.

If S is empty, $span(S) = \{\Theta\}$ – a subspace of V.

If $S \neq \emptyset$ and $u, v \in span(S)$, we can choose vectors $v_1, \dots, v_n \in S$ so that $u = a_1 v_1 + \cdots + a_n v_n$ and $v = b_1 v_1 + \dots + b_n v_n$ for some $a_i, b_i \in \mathbb{K}$. Clearly, $u + v = a_1 v_1 + \dots + a_n v_n + b_1 v_1 + \dots + b_n v_n = (a_1 + b_1) v_1 + \dots + (a_n + b_n) v_n \in W$. For every scalar $p, pu = p(a_1 v_1 + \dots + a_n v_n) = (pa_1) v_1 + \dots + (pa_n) v_n \in W$. QED

Theorem. (alternate definition of *span*)
Let $S \subseteq V$. Then $span(S)$ is the smallest subspace of V containing S.
Proof.
Every subspace of V containing S must contain all linear combinations of vectors from S. QED

We call $span(S)$ the *subspace* (*of* V) *spanned by S*.

One advantage of this definition over the other one is that it covers the case $S = \emptyset$ without branching.

Fact.

Let $V(S)$ denote the set of all subspaces of V containing S. Then

$$span(S) = \bigcap_{T \in V(S)} T$$

Proof. It is enough to show that intersection of a collection of subspaces is a subspace of V, which is easy. (Each contains Θ so the intersection does, too, etc.). QED

Examples.

In $\mathbb{C}$ over $\mathbb{R}$:

$span(\{1, i\}) = \{a \cdot 1 + b \cdot i | a, b \in \mathbb{R}\} = \mathbb{C}$

$span(\{1 + i, 2 + i\}) = \mathbb{C}$

$span(\{i + 2, 2i + 4\}) = \{2a + ai : a \in \mathbb{R}\} \neq \mathbb{C}$

In $\mathbb{R}[x]$ over $\mathbb{R}$:

$span(\{x^2, x, 1\}) = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$ – the set of all polynomials of degree at most 2.

In 2^X over $\mathbb{Z}_2$:

$span(\{A, B\}) = \{\emptyset, A, B, A \div B\}$ because $A \div (A \div B) = (A \div A) \div B = \emptyset \div \mathrm{B} = B$.

Theorem (Properties of *span*)

Let V be a vector space over a field $\mathbb{K}$ and let $S, T \subseteq$ V. Then

1. $S \subseteq span(S)$
2. $span\big(span(S)\big) = span(S)$
3. $S \subseteq T \Rightarrow span(S) \subseteq span(T)$
4. $(\forall v \in V)(v \in span(S) \Leftrightarrow span(S) = span(S \cup \{v\})$

Proof.

1. Obvious: every vector v from S is a linear combination of vectors from S. E.g. $v = 1 \cdot v$.

2. From 1., $span(S) \subseteq span\big(span(S)\big)$. To verify that $span\big(span(S)\big) \subseteq span(S)$ it is enough to notice that a linear combination of linear combinations of vectors from S is itself a linear combination of vectors from S.

3. $S \subseteq T \Rightarrow span(S) \subseteq span(T)$

4. $(\forall v \in V)(v \in span(S) \Leftrightarrow span(S) = span(S \cup \{v\})$

Proof (cont'd).

3. Obvious: since $S \subseteq T$, a linear combination of vectors from S is automatically a linear combination of vectors from T.

4. ($\Rightarrow$) From 3., $span(S) \subseteq span(S \cup \{v\})$. Suppose $w \in span(S \cup \{v\})$ i.e. $w = av + b_1u_1 + \cdots + b_ku_k$ for some $u_1, \ldots, u_k \in S$ and some scalars $b_1, \ldots, b_k$. $v \in span(S)$ means $v = c_1v_1 + \cdots + c_nv_n$. Hence, $w = a(c_1v_1 + \cdots + c_nv_n) + b_1u_1 + \cdots + b_ku_k = (ac_1)v_1 + \cdots + (ac_n)v_n + b_1u_1 + \cdots + b_ku_k \in span(S)$.

($\Leftarrow$) $span(S) = span(S \cup \{v\})$ means, in particular, that every vector from $span(S \cup \{v\})$, including v, belongs to $span(S)$.
QED